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# $PG(4,2)-\{*\}$ 上のハミルトンサイクル

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## A Hamilton cycle of $PG(4,2)-\{*\}$

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**Synopsis**

We construct a Hamilton cycle on the point-line graph of $PG(4,2)-\{*\}$.

## 1 Introduction

In this article, we prove the following theorem.

**Theorem 1.** *There exist fifteen lines $L_1, L_2, \cdots, L_{15}$ and a point $*$ in $PG(4,2)$ such that $PG(4,2)-\{*\} = L_1 \cup L_2 \cup \cdots \cup L_{15}$ with $L_i \cap L_j = \emptyset$ if $i-j \pmod{15} \geq 2$ and $j-i \pmod{15} \geq 2$.*

We define the points $P_i := L_i \cap L_{i+1}$ for $1 \leq i \leq 15$. Then this theorem says that

$$PG(4,2) = \overline{P_1P_2} \cup \overline{P_2P_3} \cup \cdots \cup \overline{P_{13}P_{14}} \cup \overline{P_{14}P_{15}} \cup \overline{P_{15}P_1} \cup \{*\},$$

where $\overline{P_iP_{i+1}} \cap \overline{P_jP_{j+1}} = \emptyset$ if and only if $\{i, i+1\} \cap \{j, j+1\} = \emptyset$.

Let $\omega$ be an element of $GF(2^2)$ which satisfies that $\omega^2+\omega+1=0$.

**Lemma 1.** *We regard $GF(2^4)-\{0\}$ as $GF(2^2)\times GF(2^2)-\{(0,0)\}$. Then we have $GF(2^4)-\{0\} = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$, where $|M_i| = 3$ with $\sum_{x\in M_i} x = 0$ for $1 \leq i \leq 5$ and $M_j \cap M_k = \emptyset$ for $1 \leq j < k \leq 5$ as follows;*

$$\begin{aligned}
M_1 &= \{a=(1,0), a\omega=(\omega,0), a\omega^2=(\omega^2,0)\},\\
M_2 &= \{b=(0,1), b\omega=(0,\omega), b\omega^2=(0,\omega^2)\},\\
M_3 &= \{c=(1,1), c\omega=(\omega,\omega), c\omega^2=(\omega^2,\omega^2)\},\\
M_4 &= \{d=(1,\omega), d\omega=(\omega,\omega^2), d\omega^2=(\omega^2,1)\},\\
M_5 &= \{e=(\omega,1), e\omega=(\omega^2,\omega), e\omega^2=(1,\omega^2)\}.
\end{aligned}$$

*We also have another decomposition $GF(2^4)-\{0\} = N_1 \cup N_2 \cup N_3 \cup N_4 \cup N_5$, where $|N_i| = 3$ with $\sum_{x\in N_i} x = 0$ for $1 \leq i \leq 5$ and $N_j \cap N_k = \emptyset$ for $1 \leq j < k \leq 5$ as follows;*

$$\begin{aligned}
N_1 &= \{a=(1,0), e=(\omega,1), d\omega^2=(\omega^2,1)\}\\
N_2 &= \{a\omega=(\omega,0), e\omega=(\omega^2,\omega), d=(1,\omega)\}\\
N_3 &= \{b=(0,1), d\omega=(\omega,\omega^2), c\omega=(\omega,\omega)\}\\
N_4 &= \{b\omega=(0,\omega), c=(1,1), e\omega^2=(1,\omega^2)\}\\
N_5 &= \{b\omega^2=(0,\omega^2), c\omega^2=(\omega^2,\omega^2), a\omega^2=(\omega^2,0)\}.
\end{aligned}$$

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**Lemma 2.** *$PG(4,2) = M_1^{'} \cup M_2^{'} \cup M_3^{'} \cup M_4^{'} \cup M_5^{'}$, where $M_i^{'} = PG(2,2)$ for $1 \le i \le 5$ such that $M_j^{'} \cap M_k^{'} = \{*\}$ for $1 \le j < k \le 5$.*

*Proof.* We regard $PG(4,2) = GF(2^4) \times GF(2) - \{(0,0)\}$ and $* = (0,1)$. ¿From $M_1, M_2, \ldots, M_5$ of Lemma 1, we define the projective planes $M_1^{'}, M_2^{'}, \ldots, M_5^{'}$ as follows;

$$\begin{aligned}
M_1^{'} &= \{(a,1),(a\omega,1),(a\omega^2,1),(a,0),(a\omega,0),(a\omega^2,0),(0,1)\} \\
M_2^{'} &= \{(b,1),(b\omega,1),(b\omega^2,1),(b,0),(b\omega,0),(b\omega^2,0),(0,1)\} \\
M_3^{'} &= \{(c,1),(c\omega,1),(c\omega^2,1),(c,0),(c\omega,0),(c\omega^2,0),(0,1)\} \\
M_4^{'} &= \{(d,1),(d\omega,1),(d\omega^2,1),(d,0),(d\omega,0),(d\omega^2,0),(0,1)\} \\
M_5^{'} &= \{(e,1),(e\omega,1),(e\omega^2,1),(e,0),(e\omega,0),(e\omega^2,0),(0,1)\}
\end{aligned}$$

Then $M_1^{'} \cup M_2^{'} \cup M_3^{'} \cup M_4^{'} \cup M_5^{'}$ is a desired decomposition. □

¿From $N_1, N_2, \ldots, N_5$ of Lemma 1, we define the lines $N_1^{'}, N_2^{'}, \ldots, N_5^{'}$ of $PG(4,2)$ as follows;

$$\begin{aligned}
N_1^{'} &= \{(a,1),(e,0),(d\omega^2,1)\} \\
N_2^{'} &= \{(a\omega,1),(e\omega,0),(d,1)\} \\
N_3^{'} &= \{(b,1),(d\omega,0),(c\omega,1)\} \\
N_4^{'} &= \{(b\omega,1),(c,0),(e\omega^2,1)\} \\
N_5^{'} &= \{(b\omega^2,1),(c\omega^2,0),(a\omega^2,1)\}.
\end{aligned}$$

## 2 Proof of Theorem 1

Let us observe the following arrangement of the sets of Lemma 1;

$$\begin{aligned}
M_1 &= \{a\omega, a\omega^2, a\}, \\
N_1 &= \{a, e, d\omega^2\}, \\
M_4 &= \{d\omega^2, d, d\omega\}, \\
N_3 &= \{d\omega, c\omega, b\}, \\
M_2 &= \{b, b\omega, b\omega^2\}, \\
N_5 &= \{b\omega^2, a\omega^2, c\omega^2\}, \\
M_3 &= \{c\omega^2, c\omega, c\}, \\
N_4 &= \{c, b\omega, e\omega^2\}, \\
M_5 &= \{e\omega^2, e, e\omega\}, \\
N_2 &= \{e\omega, d, a\omega\}.
\end{aligned}$$

Then we notice that there exists an circular arrangement of the sets

$$\cdots \to M_1 \to N_1 \to M_4 \to N_3 \to M_2 \to N_5 \to M_3 \to N_4 \to M_5 \to N_2 \to M_1 \to \cdots . \qquad (1)$$

Using (1), we define the circular arrangement of the projective planes and lines as follows;

$$\cdots \to M_1^{'} \to N_1^{'} \to M_4^{'} \to N_3^{'} \to M_2^{'} \to N_5^{'} \to M_3^{'} \to N_4^{'} \to M_5^{'} \to N_2^{'} \to M_1^{'} \to \cdots .$$

Since $PG(4,2) - \{*\} = GF(2^4) \times GF(2) - \{(0,0),(0,1)\}$, we can define the lines $L_1, L_2, \ldots, L_{15}$ using this circular arrangement.

Firstly, choose the lines $L_1$ and $L_2 \subset M_1^{'}$ as follows;

$$L_1 := \{(a\omega, 1), (a, 0), (a\omega^2, 1)\}$$
$$L_2 := \{(a\omega^2, 1), (a\omega, 0), (a, 1)\}.$$

Next we choose $L_3 := N_1^{'} = \{(a, 1), (e, 0), (d\omega^2, 1)\}$. Secondly, let the lines $L_4, L_5 \subset M_4^{'}$ as follows;

$$L_4 := \{(d\omega^2, 1), (d\omega, 0), (d, 1)\}$$
$$L_5 := \{(d, 1), (d\omega^2, 0), (d\omega, 1)\}.$$

Then choose $L_6 := N_3^{'} = \{(d\omega, 1), (c\omega, 0), (b, 1)\}$. Let the lines $L_7, L_8 \subset M_2^{'}$ as follows;

$$L_7 := \{(b, 1), (b\omega^2, 0), (b\omega, 1)\}$$
$$L_8 := \{(b\omega, 1), (b, 0), (b\omega^2, 1)\}.$$

We set $L_9 := N_5^{'} = \{(b\omega^2, 1), (a\omega^2, 0), (c\omega^2, 1)\}$. Let $L_{10}, L_{11} \subset M_3^{'}$ as;

$$L_{10} := \{(c\omega^2, 1), (c, 0), (c\omega, 1)\}$$
$$L_{11} := \{(c\omega, 1), (c\omega^2, 0), (c, 1)\}.$$

Then, we set $L_{12} := N_4^{'} = \{(c, 1), (b\omega, 0), (e\omega^2, 1)\}$. We define $L_{13}, L_{14} \subset M_5^{'}$ such as;

$$L_{13} := \{(e\omega^2, 1), (e\omega, 0), (e, 1)\}$$
$$L_{14} := \{(e, 1), (e\omega^2, 0), (e\omega, 1)\}.$$

Lastly, we define $L_{15} := N_2^{'} = \{(e\omega, 1), (d, 0), (a\omega, 1)\}$. By the above construction, we can easily check that the lines $L_1, L_2, \cdots, L_{15}$ satisfies the desired conditions.

## 文　　献

1) R. Lidl and H. Niederreiter, Finite Fields, Cambridge University press(2003).