Configuration space with labels in a partial abelian monoid

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Synopsis

G.Segal's approximation theorem asserts that the configuration space with labels in a path-connected space has the homotopy type of iterated loop-suspension of the space. In this paper we give a definition of the configuration space with labels in a partial abelian monoid. A generalization of the Segal's approximation theorem is stated as a conjecture and some partial results are given.

1 Introduction

G.Segal's approximation theorem asserts that $C(\mathbb{R}^n, X)$ is weakly homotopy equivalent to $\Omega^n\Sigma^n X$ where $C(\mathbb{R}^n, X)$ denote the configuration space of finite points in $\mathbb{R}^n$ with labels in X. In this paper, we give a definition of the configuration space with labels in a partial abelian monoid and propose a conjecture which generalizes the Segal's approximation theorem.

For this purpose, the first three sections are devoted to investigate the properties of partial abelian monoids from the homotopy theoretical point of view. More precisely, the definition of partial abelian monoid is given in §2. In §3, we give show that many functors defined on the category of topological spaces can be extended to the category of partial abelian monoids. Such functors include the variants of the loop space functors and the classifying space functor. In §4, we settle the notation for the configuration space with labels in a partial abelian monoid and investigate some of its properties. We also state a conjecture which is the generalization of the Segal's theorem to partial abelian monoid case and indicate a direction of the solution.

2 The category of partial abelian monoids

The notion of partial monoid appeared in a work of G.Segal seems the first use of such a notion in algebraic topology.[6] Later, more general definition of such a notion was given by Shimakawa. [7] In this section, we introduce one form of such a notion adapted to the purpose of this paper. We restrict ourselves to the abelian case. Remark that our definition of partial abelian monoid is a special case of that given by Shimakawa.

Definition 2.1. A space M is a (topological) partial abelian monoid if there exist a subspace $M_2 \subset M \times M$ and a map $m : M_2 \to M$ such that

(1) $M \vee M \subset M_2$,

(2) $(a, b) \in M_2$ if and only if $(b, a) \in M_2$,

(3) $(m(a, b), c) \in M_2$ if and only if $(a, m(b, c)) \in M_2$,

(4) $m(a, *_M) = m(*_M, a) = a$,

(5) $m(a, b) = m(b, a)$, and

(6) $m(m(a, b), c) = m(a, m(b, c))$.

We write $m(a, b) = a + b$. An element in M_2 is called a summable pair. Let M_k be a subset of M^k which consists of those k-tuples $(a_1, \cdots, a_k)$ such that $a_1 + \cdots + a_k$ is defined.

Definition 2.2. A map $f : M \to N$ between partial abelian monoids M, N is called partial abelian monoid homomorphism if $f \times f$ maps M_2 into N_2 and $f(a + b) = f(a) + f(b)$. We denote the category of partial abelian monoids and partial abelian monoid homomorphisms by $\mathcal{PAM}$.

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Example 2.3. Any space X is regarded as a partial abelian monoid by a subspace $X_2 = X \vee X \subset X^2$ and the folding map. We call this structure a trivial partial abelian monoid structure on X.

Example 2.4. Let M be an abelian monoid and N be a subspace of M. Then N has a partial abelian monoid structure if we put $N_2 = \{(a, b) | a + b \in N\}$.

Example 2.5. Contrary to the above example, we restrict the sum itself to give a new partial abelian monoid from an abelian monoid. More precisely, let M be an abelian monoid. Let N be a space homeomorphic to M and N_2 be a subspace of $N \times N$ which satisfies the conditions (1)$\sim$ (3) of Definition 2.1.

3 Extension of functors defined on $\mathcal{T}op_*$

In this section, we show that many functors defined on $\mathcal{T}op_*$ can be extended to functors on $\mathcal{PAM}$. Results are stated as proposition, but the proof of functoriality is easy and left to the reader.

3.1 Reduced suspension functor Σ

Let ΣM denote the reduced suspension space $\Sigma M = S^1 \wedge M$. Let $s \wedge a$ and $t \wedge b$ be elements of ΣM. We say that $s \wedge a$ and $t \wedge b$ is summable if $s = t$ and a and b are summable as elements in M. The sum is given by $s \wedge a + s \wedge b = s \wedge (a + b)$.

Proposition 3.1. *The suspension functor $(- \wedge -)$ is a self-functor on $\mathcal{PAM}$.*

Suspension functor can be viewed as an application of the bivariant functor $(- \wedge -)$; $\Sigma M \approx S^1 \wedge M$ where S^1 is considered as a partial abelian monoid with the trivial partial abelian monoid structure.

3.2 Smash product $(- \wedge -)$

As a space, smash product $M \wedge N$ of M and N is the quotient space $M \times N / M \vee N$ where $M \vee N$ is identified with $\{(a, 0)\} \cup \{(0, b)\} \subset M \times N$. A class represented by (a, b) is denoted by $a \wedge b \in M \wedge N$, where $a \in M$ and $b \in N$. We give a partial abelian monoid structure on $M \wedge N$ by considering $a \wedge b$ and $c \wedge d$ are summable if one of the following holds :

(1) $b = d$ and $(a, c) \in M_2$, or

(2) $a = c$ and $(b, d) \in N_2$.

The sum of such elements is given by

$$a \wedge b + c \wedge d = \begin{cases} (a + c) \wedge b = (a + c) \wedge d & \text{if } b = d \text{ and } (a, c) \in M_2 \\ a \wedge (b + d) = c \wedge (b + d) & \text{if } a = c \text{ and } (b, d) \in N_2. \end{cases}$$

Proposition 3.2. *The smash product functor $(- \wedge -)$ is a functor from the product category $\mathcal{PAM} \times \mathcal{PAM}$ to $\mathcal{PAM}$. It is covariant on both variables.*

3.3 Mapping space $\mathrm{Map}_*(-, -)$

As a space, the mapping space $\mathrm{Map}_*(M, N)$ of M and N is the space of based maps from M to N. We give a partial abelian monoid structure on $\mathrm{Map}_*(M, N)$ by considering f and g in $\mathrm{Map}_*(M, N)$ is summable if $f(a)$ and $g(a)$ are summable in N for all $a \in M$. The sum of such maps is given by defining the map $f + g$ by the map

$$(f + g)(a) = f(a) + g(a).$$

Proposition 3.3. *The mapping space functor $Map_*(-, -)$ is a functor from the product category $\mathcal{PAM} \times \mathcal{PAM}$ to $\mathcal{PAM}$. It is contravariant on the first variable and covariant on the second variable.*

3.4 Loop space functor and its variants

Let $\Omega M = \mathrm{Map}_*(S^1, M)$ be the space of based loops on M, thinking of M as a space. Partial abelian monoid structure of M induces a partial abelian monoid structure on $\Omega M = \mathrm{Map}_*(S^1, M)$.

$\Omega M = \mathrm{Map}_*(S^1, M)$ is a partial abelian monoid by putting

$$(\Omega M)_2 = \{(\lambda, \mu) \mid (\lambda(t), \mu(t)) \in M_2 \text{ for all } t \in S^1\}$$

and

$$(\lambda + \mu)(t) = \lambda(t) + \mu(t).$$

Proposition 3.4. Ω *is a self-functor on* $\mathcal{PAM}$.

It is useful to introduce Moore loop space when we want a strict monoid rather than a partial monoid on the loop space. Let $\Omega_s(M)$ denote the space of loops of length s, that is,

$$\Omega_s(M) = \{l : [0,s] \to M \mid l(0) = l(s) = *_M\}.$$

We give a partial abelian monoid structure on $\Omega_s(M)$ by putting

$$(\Omega_s(M))_2 = \{(l_1, l_2) \mid (l_1(t), l_2(t)) \in M_2 \text{ for all } t \in [0,s]\}$$

and

$$(l_1 + l_2)(t) = l_1(t) + l_2(t) \in \Omega_s(M),\ t \in [0,s].$$

Proposition 3.5. Ω_s *is a self-functor on* $\mathcal{PAM}$.

The Moore loop space is defined by $\Lambda M = \bigcup_{s\geq 0} \Omega_s(M) \times \{s\}$. We can identify $\Omega_s(M)$ as a subspace of $\mathrm{Map}([0,\infty), M)$ by extending $l : [0,s] \to M$ to $[0,\infty)$ by $l(t) = *$ for any $t \in (s,\infty)$. We topologize ΛM as a subspace of $\mathrm{Map}([0,\infty), M) \times [0,\infty)$ by this identification.

ΛM also have a structure of partial abelian monoid . Let (α, s) and (β, t) be elements of ΛM, where $\alpha : [0,s] \to M$ and $\beta : [0,t] \to M$. (α, s) and (β, t) are summable if $s = t$ and α and β are summable as elements in $\Omega_s M$.

Proposition 3.6. Λ *is a self-functor on* $\mathcal{PAM}$.

3.5 Classifying space functor B

For any partial abelian monoid M we have a simplicial space M_* given by $M_0 = *, M_1 = M$ and

$$M_k = (\text{the space of composable } k\text{-tuples}) \text{ for } k \geq 2.$$

The structure maps of the simplicial space is $s_i : M_k \to M_{k+1}$ $(0 \leq i \leq k)$ given by

$$s_i(a_1, \cdots, a_k) = (a_1, \cdots, a_i, 0, a_{i+1}, \cdots, a_k)$$

and $d_i : M_k \to M_{k-1}$ $(0 \leq i \leq k)$ given by

$$d_0(a_1, \cdots, a_k) = (a_2, \cdots, a_k)$$

$$d_i(a_1, \cdots, a_k) = (a_1, \cdots, a_i + a_{i+1}, \cdots, a_k)$$

and

$$d_k(a_1, \cdots, a_k) = (a_1, \cdots, a_{k-1}).$$

The classifying space of M, denoted by BM, is defined as the geometric realization of the above simplicial space. Suppose $[c,m], [c',m'] \in BM$ and $(m(t), m'(t)) \in M_2$ for all $t \in c \cap c'$. We define a map $m \cup m' : c \cup c' \to M$ by

$$(m \cup m')(t) = \begin{cases} m(t) & \text{if } t \in c - c' \\ m'(t) & \text{if } t \in c' - c \\ m(t) + m'(t) & \text{if } t \in c \cap c'. \end{cases}$$

Then we $[c,m]$ and $[c',m']$ are summable if the set $\{(m \cup m')(t) \mid t \in c \cup c'\}$ is a summable set. (We say that a finite subset A of a partial abelian monoid M is a summable set if $(a_1, cdots, a_k)$ is a summable k-tuple, where $\#A = k$ and $a_1, \cdots, a_k$ are distinct elements of A. This definition does not depend on the choice of the ordering of $a_1, \cdots, a_k$ since our partial monoids are abelian.) Thus the partial abelian monoid structure on BM is given by

$$(BM)_2 = \left\{ ([c,m],[c',m']) \;\middle|\; \begin{array}{l} (m(t), m'(t)) \in M_2 \text{ for any } t \in c \cap c' \\ \{(m \cup m')(t) | t \in c \cup c'\} \text{ is a summable set} \end{array} \right\}.$$

Suppose that $([c,m],[c',m']) \in (BM)_2$. Then we put

$$[c,m] + [c',m'] = [c \cup c', m \cup m'].$$

Proposition 3.7. *B is a self-functor on $\mathcal{PAM}$.*

When M is a monoid, ΣM is identified with the 1-skelton of BM. This also applies to our situation. Furthermore ΣM is a sub-partial abelian monoid of BM. We have an inclusion $\iota : \Sigma M \to BM$ defined by the correspondence $t \wedge a \to [t, a]$, where $t \in [0, 1]$ and $a \in M$. If $t \wedge a$ and $u \wedge b$ are summable in ΣM, we have $t = u$ and $(a, b) \in M_2$. Then $(\iota(t \wedge a), \iota(t \wedge b)) = ([t, a], [t, b]) \in (BM)_2$, which shows that $(\iota \times \iota)(M_2) \subset (BM)_2$. For the same pair $(t \wedge a, t \wedge b)$, we have $\iota(t \wedge a + t \wedge b) = \iota(t \wedge (a + b)) = [t, a + b]$. On the other hand $\iota(t \wedge a) + \iota(t \wedge b) = [t, a] + [t, b] = [t, a + b]$, which shows that ι is a partial abelian monoid homomorphism.

4 Configuration space with summable labels

Salvatore considered the configuration space with labels in a non-commutative partial monoid.[5)] Using a Fulton-MacPherson operad action, he showed that n-monoid is equivalent to an n-fold loop space. On the other hand, Shimakawa introduced the configuration space with labels in a partial abelian monoid and showed that if we take the infinite limit of the domain space $\mathbb{R}^n$, this leads to a several homology theories after taking the homotopy groups. [7)]. The notion of the configuration space with summable labels are given by Kallel. [1)],[2)],[3)]

The definition given below is a quite special case of the preceding models, but it is very simple and is sufficient for our purpose. Especially we restrict ourselves to the two-generated and abelian case.

Definition 4.1. Let $Z_n^{(k)}(M)$ denote the subspace of $(\mathbb{R}^n \times M)^k$ given by

$$Z_n^{(k)}(M) = \left\{ ((v_1, a_1), \cdots, (v_k, a_k)) \;\middle|\; \begin{array}{l} \text{for any } i_1, \cdots, i_r \text{ such that} \\ v_{i_1} = \cdots = v_{i_r}, \\ (a_{i_1}, \cdots, a_{i_r}) \in M_r \end{array} \right\}.$$

Then we define a space $C(\mathbb{R}^n, M)$ as $C(\mathbb{R}^n, M) = (\coprod_{k \geq 0} Z_n^{(k)}(M))/\sim$, where $\sim$ denotes the least equivalence relation which satisfies (R1)~(R3) below.

(R1) If $a_i = *_M$ then

$$((v_1, a_1), \cdots, (v_k, a_k)) \sim ((v_1, a_1), \cdots, \widehat{(v_i, a_i)}, \cdots, (v_k, a_k)).$$

(R2) For any permutation $\sigma \in \Sigma_k$,

$$((v_1, a_1), \cdots, (v_k, a_k)) \sim ((v_{\sigma^{-1}(1)}, a_{\sigma^{-1}(1)}), \cdots, (v_{\sigma^{-1}(k)}, a_{\sigma^{-1}(k)})).$$

(R3) If $v_1 = v_2 = v$,

$$((v_1, a_1), \cdots, (v_k, a_k)) \sim ((v, a_1 + a_2), (v_3, a_3), \cdots, (v_k, a_k)).$$

We regard $C(\mathbb{R}^n, M)$ as a partial abelian monoid as follows. Let $C(\mathbb{R}^n, M)_2 \subset C(\mathbb{R}^n, M)^2$ be the subspace which consists of pairs (ξ, η) such that $(a_i, b_i) \in M_2$ for every i where

$$\xi = [(v_1, a_1), \cdots, (v_k, a_k)] \text{ and } \eta = [(v_1, b_1), \cdots, (v_k, b_k)].$$

Then $m : C(\mathbb{R}^n, M)_2 \to C(\mathbb{R}^n, M)$ is defined by setting

$$m(\xi, \eta) = [(v_1, a_1 + b_1), \cdots, (v_k, a_k + b_k)].$$

Proposition 4.2. *The functor $C(\mathbb{R}^n, -)$ is a functor from $\mathcal{PAM}$ to itself.*

Example 4.3. Recall from Example 2.3 that any space X has a partial abelian monoid structure called the trivial partial sum given by a subspace $X_2 = X \vee X \subset X^2$ and the folding map. Then $C(\mathbb{R}^n, X)$ is nothing but the configuration space of finite points in $\mathbb{R}^n$ labeled by X.

Example 4.4. Let $M = S^0 = \{0, 1\}$ with the trivial partial abelian monoid structure. Then $C(\mathbb{R}^n, S^0)$ is homeomorphic to the configuration space of particles in $\mathbb{R}^n$.

Example 4.5. Let $M = \{-1, 0, 1\}$ with $1 + (-1) = 0$ as the only non-trivial partial sum. Then $C(\mathbb{R}^n, M)$ is homeomorphic to the configuration space of positive and negative particles in $\mathbb{R}^n$.

Lemma 4.6 below says that $C(\mathbb{R}^n, -)$ is a homotopy functor in some sense. Let M be a partial abelian monoid. We regard $M \times I$ as a partial abelian monoid by setting

$$(M \times I)_2 = \{(m, t; n, t) \mid (m, n) \in M_2\}$$

and

$$\mu_{M \times I}(m, t; n, t) = (\mu_M(m, n), t).$$

Homomorphisms $f, g : M \to N$ between partial abelian monoids are called homotopic via partial abelian monoid homomorphisms if there exists a homomorphism $H : M \times I \to N$ such that $H_0 = f, H_1 = g$.

Lemma 4.6. *If $f, g : M \to N$ are homotopic via partial abelian monoid homomorphisms then $C(\mathbb{R}^n, f)$ and $C(\mathbb{R}^n, g)$ are homotopic via partial abelian monoid homomorphisms.*

Proof. Let $H : M \times I \to N$ be a homotopy between f and g. Observe that we have a homomorphism

$$\varphi : C(\mathbb{R}^n, M) \times I \to C(\mathbb{R}^n, M \times I)$$

defined by the correspondence

$$([(v_1, a_1), \cdots, (v_k, a_k)], t) \mapsto [(v_1, (a_1, t)), \cdots, (v_k, (a_k, t))].$$

Consider the composite

$$C(\mathbb{R}^n, M) \times I \xrightarrow{\varphi} C(\mathbb{R}^n, M \times I) \xrightarrow{C(\mathbb{R}^n, H)} C(\mathbb{R}^n, N).$$

Then

$$\begin{aligned} &C(\mathbb{R}^n, H) \circ \varphi(([(v_1, a_1), \cdots, (v_k, a_k)], t)) \\ &= C(\mathbb{R}^n, H)([(v_1, (a_1, t)), \cdots, (v_k, (a_k, t))]) \\ &= [(v_1, H(a_1, t)), \cdots, (v_k, H(a_k, t))]. \end{aligned}$$

Thus we see that this composition is a homotopy between $C(\mathbb{R}^n, f)$ and $C(\mathbb{R}^n, g)$ via partial abelian monoid homomorphisms. □

Proposition 4.7. *$BC(\mathbb{R}^n, M)$ is homeomorphic to $C(\mathbb{R}^n, BM)$.*

Proof. Note that any element of $BC(\mathbb{R}^n, M)$ can be considered as a configuration of points in $[0, 1]$ labelled by points in $C(\mathbb{R}^n, M)$. So we may assume that the representative is of the form $(t_1, \xi_1; \cdots; t_k, \xi_k)$ where $t_1, \cdots, t_k$ are real numbers such that $0 \le t_1 \le \cdots \le t_k \le 1$ and ξ_i are elements of $C(\mathbb{R}^n, M)$ A map $\varphi : BC(\mathbb{R}^n, M) \to C(\mathbb{R}^n, BM)$ is defined by the correspondence

$$[t_1, \xi_1; \cdots; t_k, \xi_k] \mapsto \sum_{i=1}^{k} [v_{i1}, (t_i, a_{i1}); \cdots; v_{ir_i}, (t_i, a_{ir_i})]$$

where $0 \le t_1 \le t_2 \le \cdots \le t_k \le 1$,

$$\xi_i = [v_{i1}, a_{i1}; \cdots; v_{ir_i}, a_{ir_i}] \in C(\mathbb{R}^n, M)$$

and $v_{is} \in \mathbb{R}^n, a_{is} \in M$. We define a map $\psi : C(\mathbb{R}^n, BM) \to BC(\mathbb{R}^n, M)$ by the correspondence

$$[v_1, \beta_1; \cdots; v_l, \beta_l] \mapsto \sum_{i=1}^{l} [t_{i1}, [v_i, a_{i1}]; \cdots; t_{is_i}, [v_i, a_{is_i}]],$$

where $\beta_i = [t_{i1}, a_{i1}; \cdots; t_{is_i}, a_{is_i}]$. We have

$$\psi \circ \varphi([t_1, \xi_1; \cdots; t_k, \xi_k]) = \psi(\sum_{i=1}^{k} ([v_{i1}, [t_i, a_{i1}]; \cdots; v_{ir_i}, [t_i, a_{ir_i}]])$$

$$= \sum_{i=1}^{k} \psi([v_{i1}, [t_i, a_{i1}]; \cdots; v_{ir_i}, [t_i, a_{ir_i}]]) = \sum_{i=1}^{k} [t_i, \xi_i],$$

which shows that $\psi \circ \varphi$ is the identity on $BC(\mathbb{R}^n, M)$. On the other hand we have

$$\varphi \circ \psi([v_1, \beta_1; \cdots; v_l, \beta_l]) = \varphi(\sum_{i=1}^{l}[t_{i1}, [v_i, a_{i1}]; \cdots; t_{is_i}, [v_i, a_{is_i}]])$$

$$= \sum_{i=1}^{l}\sum_{j=1}^{is_j}[v_i, [t_{ij}, a_{ij}]] = \sum_{i=1}^{l}[v_i, \beta_i]$$

which shows that $\varphi \circ \psi$ is the identity on $C(\mathbb{R}^n, BM)$. □

This is a generalization of a proposition given by Segal [6] in the sense that if $M = X$ is a space with trivial partial abelian monoid structure then $BM = \Sigma X$ and the homeomorphism specializes to $BC(\mathbb{R}^n, X) \approx C(\mathbb{R}^n, \Sigma X)$.

By the above proposition, we know that B and $C(\mathbb{R}^n, -)$, considered as self-functors on $\mathcal{PAM}$, are commutative. We can see that $C(\mathbb{R}^n, -)$ behaves well among other self-functors introduced in §2.

We consider the subspace $SC(\mathbb{R}^n, M) \subset BC(\mathbb{R}^n, M)$ consisting of all the elements $[t_1, \xi_1; \ldots; t_k, \xi_k]$ where $\xi_i = [v_{i1}, a_{i1}, \ldots, v_{ir_i}, a_{ir_i}]$ is an element of $C(\mathbb{R}^n, M)$ such that v_{ij} are distinct. Then by a careful check of the proof of Proposition 4.7 , we have the following corollary.

Corolary 4.8. *$SC(\mathbb{R}^n, M)$ is homeomorphic to $C(\mathbb{R}^n, \Sigma M)$.*

The relation between $C(\mathbb{R}^n, -)$ and loop space functors can be stated as follows.

Proposition 4.9. *There exist natural transformations of functors*

$$C(\mathbb{R}^n, -) \circ \Omega \Rightarrow \Omega \circ C(\mathbb{R}^n, -)$$

and

$$C(\mathbb{R}^n, -) \circ \Omega_s \Rightarrow \Omega_s \circ C(\mathbb{R}^n, -).$$

Proof. We give $\phi_M : C(\mathbb{R}^n, \Omega M) \to \Omega C(\mathbb{R}^n, M)$ by

$$[(v_1, l_1), \cdots, (v_k, l_k)] \mapsto (t \mapsto [(v_1, l_1(t)), \cdots, (v_k, l_k(t))])$$

where $v_i \in \mathbb{R}^n$ and $l_i \in \Omega M$. Next, we show that the following diagram commutes:

$$\begin{array}{ccc} C(\mathbb{R}^n, \Omega M) & \xrightarrow{\phi_M} & \Omega C(\mathbb{R}^n, M) \\ {\scriptstyle C(\mathbb{R}^n, \Omega f)}\downarrow & & {\scriptstyle \Omega C(\mathbb{R}^n, f)}\downarrow \\ C(\mathbb{R}^n, \Omega N) & \xrightarrow{\phi_N} & \Omega C(\mathbb{R}^n, N), \end{array}$$

where $f : M \to N$ is a partial abelian monoid homomorphism. Observe that the image of $[(v_1, l_1), \cdots, (v_k, l_k)] \in C(\mathbb{R}^n, \Omega M)$ by $C(\mathbb{R}^n, f) \circ \phi_M$ is a loop in $C(\mathbb{R}^n, N)$ given by

$$(t \mapsto [(v_1, f(l_1(t))), \cdots, (v_k, f(l_k(t)))])$$

and the image of the same element by $\phi_N \circ C(\mathbb{R}^n, f)$ is given by a loop

$$(t \mapsto [(v_1, (\Omega f \circ l_1)(t)), \cdots, (v_k, (\Omega f \circ l_k)(t))])$$

which coincide with each other. This shows that there exists a natural transformation

$$C(\mathbb{R}^n, -) \circ \Omega \Rightarrow \Omega \circ C(\mathbb{R}^n, -).$$

The proof applies when Ω is replaced by Ω_s. □

Proposition 4.10. *There exists a natural transformation of functors*

$$C(\mathbb{R}^n, -) \circ \Lambda \Rightarrow \Lambda \circ C(\mathbb{R}^n, -).$$

Proof. In this case, we give $\phi_M : C(\mathbb{R}^n, \Lambda M) \to \Lambda C(\mathbb{R}^n, M)$ by $\phi_M([(v_1,(l_1,s_1)),\cdots,(v_k,(l_k,s_k))]) = (l,s)$ where $s = \mathrm{Max}\{s_1,\cdots,s_k\}$ and $l : [0,s] \to C(\mathbb{R}^n, M)$ is given by $l(t) = [(v_1,l_1(t)),\cdots,(v_k,l_k(t))]$. Here, the domain of $l_i(t)$ is extended to $[0,s]$ by considering $l_i(t) = *$ for $t > s_i$. Proof that ϕ gives the natural transformation is similar to the proof of the above proposition. □

We may ask the homotopy type of $C(\mathbb{R}^n, M)$. We expect that a generalization of the Segal's approximation theorem [6] holds. In the Segal's theorem, it was assumed that X is path-connected. This theorem is well understood in the context of group completion if we say the connectivity condition as '$\pi_0(X)$ is a group.' When we attempt to generalize the Segal's theorem to partial abelian monoid M in place of X, we need the corresponding notion. The following notion is one such generalization of the connectivity condition.

Definition 4.11. A partial abelian monoid is grouplike if for any $a \in M$ there exists $b \in M$ such that $a + b \in M_0$, where M_0 denotes the path-component of 0.

If we think of the Segal's theorem as a theorem about the self-functors $C(\mathbb{R}^n, -)$ (in Segal's notation, $C_n(-)$) and $\Omega^n\Sigma^n$ on the category of based spaces, then the generalization to the partial abelian monoid case should be the theorem about the self-functors on $\mathcal{PAM}$. We propose the following conjecture.

Conjecture 4.12. *$C(\mathbb{R}^n, M)$ is homotopy equivalent to $\Omega^n B^n M$ if M is grouplike.*

By Proposition 4.7, we may expect that we can generalize the lines of the Segal's paper [6] to prove the above conjecture. But we do not know how to manage the term 'weakly homotopy equivalent as partial abelian monoid ' at this point. For example suppose we define this notion as follows :

Predefinition 4.13. Partial abelian monoids M and N are weakly homotopy equivalent if there exists a partial abelian monoid homomorphism $M \to N$ which is a weak homotopy equivalence *i.e.* there exists an inverse homomorphism of $f_* : \pi_*(M) \to \pi_*(N)$.

Then we do not know how to show the statement such as :

Preproposition 4.14. *$C(\mathbb{R}^n, M)$ is weakly homotopy equivalent to $C(\mathbb{R}^n, N)$ whenever M is weakly homotopy equivalent to N.*

It seems that we need a more sophisticated definition of the homotopy groups of partial abelian monoids or homomorphisms between them so that they reflects the partial abelian monoid structure to ensure that such a statement holds. We propose the other problems related to the above problem.

Problem 4.15. *Encode the partial abelian monoid structure properly in the homotopy group $\pi_*(M)$.*

It seems that Definition 4.11 can be stated in terms of $\pi_0(M)$ if we answer to this problem. The following problem is a very natural question, which is related to the Preproposition 4.14.

Problem 4.16. *Describe the homotopy group $\pi_*(C(\mathbb{R}^n, M))$ in terms of $\pi_*(M)$.*

References

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